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A Legendre Polynomial Integral

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Abstract. Let $\{P_n(x)\}$ be the usual Legendre polynomials. The following integral is apparently new.

$$\int_0^1 P_n(2x-1)\log\frac{1}{x}\,dx = \frac{(-1)^n}{n(n+1)} \quad \text{for } n \geq 1.$$

It has an application in the construction of Gauss quadrature formulas on (0, 1) with weight function $\log(1/x)$.

1. **Motivation.** For integrals of the type $\int_a^b f(x)w(x)\,dx$, where $w(x)$ is positive in (a, b), Gaussian quadrature formulas of the type

$$\int_a^b f(x)w(x)\,dx \approx \sum_{k=1}^{n} h_{kn} f(\xi_{kn})$$

are often useful. The $\{h_{kn}\}$ and $\{\xi_{kn}\}$ are chosen to make the formulas exact when $f(x)$ is a polynomial of degree $2n-1$ or less [1]. These formulas are especially useful when $w(x)$ is singular at one or more points in the interval.

The method of modified moments [2], [3], [4] provides a stable method for calculating the $\{h_{kn}, \xi_{kn}\}$ if the set of polynomials orthogonal on (a, b) with weight function $w(x)$ are known. That is, a set of $\{Q_k\}$, such that

$$\int_a^b Q_k(x)Q_m(x)w(x)\,dx = 0 \quad \text{if } k \neq m$$

is desired. Any such family of orthogonal polynomials obeys a three-term recurrence relation [5],

$$Q_{-1}(x) = 0, \qquad Q_0(x) = 1,$$

$$xQ_k(x) = a_kQ_{k+1}(x) + b_kQ_k(x) + c_kQ_{k-1}(x), \qquad k \geq 1,$$

with $a_k \neq 0$.

For some intervals and weight functions, the orthogonal polynomials are known, and there is no problem. For example, if $a = -1$, $b = +1$, and $w(x) = 1$, the usual Legendre polynomials $\{P_k(x)\}$ are an orthogonal set,

$$\int_{-1}^{1} P_k(x)P_m(x)\,dx = 0 \quad \text{if } k \neq m.$$

For most intervals and weight functions, the corresponding orthogonal polynomials are not known. If the moments $\int_a^b x^k w(x)\,dx$ are known, the $\{a_k, b_k, c_k\}$ of the unknown set of orthogonal polynomials can be found [2], but the process is nu-

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merically unstable [3], [4]. More generally, if $\{\overline{Q}_k\}$ is any set of polynomials, not necessarily obeying any orthogonality relation, but obeying a three-term recurrence relation

$$x\overline{Q}_k(x) = \overline{a}_k\overline{Q}_{k+1}(x) + \overline{b}_k\overline{Q}_k(x) + \overline{c}_k\overline{Q}_{k-1}(x),$$

the $\{a_k, b_k, c_k\}$ of the unknown set of orthogonal polynomials can be found [4]. For this, the modified moments $\int_a^b \overline{Q}_k(x)w(x)\,dx$ are needed. The stability of the process depends on the $\{\overline{Q}_k\}$. Some particular examples [3], [4] suggest that, for finite a and b, the process is probably stable if the $\{\overline{Q}_k\}$ are themselves orthogonal polynomials with some weight function $\overline{w}(x)$.

The appropriate orthogonal polynomials for

$$\int_0^1 f(x) \log \frac{1}{x}\,dx$$

are not known analytically. The Altran symbolic algebra package [6] was used to calculate the modified moments for various sets of orthogonal polynomials. The shifted Legendre polynomials [5], $\{P_k^*(x)\}$, with $P_k^*(x) = P_k(2x - 1)$, were found to have a particularly simple formula for modified moments, and the algorithm of [4] was found to be stable.

2. A Legendre Polynomial Integral.

THEOREM. *Let $P_n^*(x)$ be the nth shifted Legendre polynomial. Define $\nu_n = \int_0^1 P_n^*(x)\log(1/x)\,dx$. For $n \geqslant 1$, $\nu_n = (-1)^n/n(n+1)$.*

Proof. By induction. Using $P_k^*(x) = P_k(2x-1)$, from [5] we obtain

$$P_0^*(x) = 1, \qquad P_1^*(x) = 2x - 1, \qquad P_2^*(x) = 6x^2 - 6x + 1,$$

$$(k+1)P_{k+1}^*(x) = (2k+1)(2x-1)P_k^*(x) - kP_{k-1}^*(x), \qquad k \geqslant 2.$$

Note that $P_n^*(1) = 1$. The first three modified moments are $\nu_0 = 1$, $\nu_1 = -1/2$ and $\nu_2 = 1/6$. We define $\mu_n = \int_0^1 (2x-1)P_n^*(x)\log(1/x)\,dx$.

Assume $\nu_k = (-1)^k/k(k+1)$ for $k \geqslant 2$. Using the recurrence relation,

$$\nu_{k+1} = \int_0^1 P_{k+1}^*(x) \log \frac{1}{x}\,dx = \frac{1}{k+1}[(2n+1)\mu_k - k\nu_{k-1}]. \tag{1}$$

Also from [5], the derivative of $P_k^*(x)$ is

$$\frac{d}{dx}P_k^*(x) = \frac{-k}{2x(1-x)}[(2x-1)P_k^*(x) - P_{k-1}^*(x)].$$

Integrate by parts in the definition of μ_k to obtain

$$\mu_k = P_k^*(x)\left[x(1-x)\ln x + \frac{1}{2}x^2 - x\right]\Bigg|_0^1$$

$$+ \frac{k}{4}\int_0^1 \frac{x-2}{1-x}[(2x-1)P_k^*(x) - P_{k-1}^*(x)]\,dx.$$

$$- \frac{k}{2}\int_0^1 [(2x-1)P_k^*(x) - P_{k-1}^*(x)] \log \frac{1}{x}\,dx$$

Simplifying, and using $P_k^*(1) = 1$,

$$\mu_k = -\frac{1}{2} - \frac{k}{2}\mu_k + \frac{k}{2}\nu_{k-1} - \frac{1}{2}\int_0^1 x(x-2)\left[\frac{d}{dx}P_k^*(x)\right] dx.$$

The last integral may be integrated by parts, giving

$$-\frac{1}{2}x(x-2)P_k^*(x)\Big|_0^1 + 2\int_0^1 (x-1)P_k^*(x)\,dx.$$

The integrated term is $1/2$, and the integral is zero for $k > 1$ because of the orthogonality of the $\{P_k^*\}$. Thus,

$$\mu_k = \frac{k}{2}(\nu_{k-1} - \mu_k), \qquad \mu_k = \frac{k}{k+2}\,\nu_{k-1}.$$

Inserting this result in (1),

$$\nu_{k+1} = \frac{k}{k+1}\left[\frac{2k+1}{k+2} - 1\right]\nu_{k-1} = \frac{k(k-1)}{(k+1)(k+2)}\,\frac{(-1)^{k-1}}{k(k-1)}$$

$$= \frac{(-1)^{k+1}}{(k+1)(k+2)}. \;\square$$

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